

# Moving point charge as a soliton in nonlinear electrodynamics

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The field of a moving pointlike charge is determined in nonlinear local electrodynamics. As a model Lagrangian for the latter we take the one whose nonlinearity is the Euler-Heisenberg Lagrangian of quantum electrodynamics truncated at the leading term of its expansion in powers of the first field invariant. The total energy of the field produced by a point charge is finite in that model; thereby its field configuration is a soliton. We define a finite energy-momentum vector of this field configuration to demonstrate that its components satisfy the standard mechanical relation characteristic of a free moving massive particle.

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## I. INTRODUCTION

In this paper we are studying the field of a uniformly moving charge in nonlinear electrodynamics, in other words we are dealing with nonlinear extension of the Liénard-Wiechert potentials.

The problem of the field caused by a moving charge, besides belonging to fundamental problems of electrodynamics, is also of certain practical importance as applied to charged particle beams in accelerators [1]. Its nonlinear extension may have an effect in small-impact-parameter scattering at high energies where close vicinities of the charge come into play.

We take the simplest version of nonlinear electrodynamics [2], namely the one that results from keeping only the leading term in the expansion of the Heisenberg-Euler action [3] of quantum electrodynamics (QED) in powers of the field invariant  $\mathfrak{F}(x) = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(B^2 - E^2)$ , with the second invariant  $\mathfrak{G}(x) = \frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = \mathbf{B} \cdot \mathbf{E}$  set equal to zero. (Remind that, the Heisenberg-Euler action reflects the nonlinearity due to interaction between electromagnetic fields stemming from their quantum nature, and it is local in the sense that dependence on space-time derivatives of the fields present in the full effective action of QED is disregarded from it.) This nonlinear model may be thought of as an extension of the established theory of electromagnetism to the close vicinity of a point charge compatible with what that theory describes at larger distances. When treated seriously, the above truncated model results in the electric field behavior near the point charge less singular than the Coulomb law, which provides convergence of the field-energy integral [2]. This means that the field configuration produced by a point charge may be referred to as a soliton. In Section III we present the electric and magnetic fields of this soliton when it moves with a constant speed. In Section IV we define the energy-momentum of this moving soliton, which is just the energy-momentum vector of a moving particle with its mass defined by an integral of its field. It is proportional to the charge squared.

The following remark is in order. Our approach to considering the nonlinear problem

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includes as its starting point a covariant representation for the current, the field, the energy-momentum tensor and vector. In the trivial, linear limit, this approach results in totally reproducing the field of a charge moving with a constant speed as given in the Landau and Lifshitz text-book [4]. This seemingly undistinguished fact is worth stressing, however, because, strange though it may seem, one faces a discord in the present-day literature, many basic text-books included, concerning this fundamental point (see the analysis in [5]). The dispute is fueled by the expression for the field of a moving charge given without a detailed derivation in Feynman's lectures [6], that may seem to be different from that of Ref. [4], the more so that apparently different physical ideas are layed into it. This fact even called into being an experimental verification [1] that resulted in favor of [4]. For this reason we found it not out of place to analyse the result of [6] in a separate forthcoming publication [7] to reveal its complete coincidence with that of [4] both for accelerated motion of the charge and its motion with a constant speed.

## II. MINIMALLY NONLINEAR MODEL

The Lagrangian of a minimally nonlinear electrodynamics is defined as

$$L(x) = -\mathfrak{F}(x) + \frac{\sigma}{2} (\mathfrak{F}(x))^2, \quad (1)$$

where  $\mathfrak{F}(x)$  makes up (with the reversed sign) the Lagrangian density of the standard linear Maxwell electrodynamics, while the second term in (1) is the quartic in the field-strength addition to it. The field-strength tensor is related to the four-vector potential  $A^\mu(x)$  as  $F^{\mu\nu} = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)$ , where  $\partial^\mu = \frac{\partial}{\partial x_\mu}$ . Hence, the first pair of the Maxwell equations,  $[\nabla \times \mathbf{B}] = 0$  and  $[\nabla \times \mathbf{E}] + \partial^0 \mathbf{B} = \mathbf{0}$ , with the electric and magnetic field strengths  $E_i = F^{i0}$  and  $B_i = \epsilon_{ijk} F^{jk}$ , remains standard. The self-coupling constant  $\sigma$  is presumably small enough. It has the dimensionality of inverse fourth power of mass [13]. The four-dimensional scalar product is  $(ux) = u_0 x_0 - (\mathbf{u} \cdot \mathbf{x})$ ,  $x^2 = x_0^2 - |\mathbf{x}|^2$ . We use Lorentz-Heaviside system of units. If the quartic correction in (1) is understood as the lowest term in the expansion of the Heisenberg-Euler Lagrangian in powers of  $\mathfrak{F}$ , with  $\mathfrak{G} = 0$ , the value [8] of the photon selfcoupling constant is  $\sigma = e^4 \hbar / (45 \pi^2 m^4 c^7)$ , where  $e$  and  $m$  are electron charge and mass, respectively.

The action includes the selfinteraction of the electromagnetic field and interaction with a current  $j^\nu$ , produced by a moving charge. It is

$$S[A] = -\frac{1}{c} \int \left( \frac{1}{c} A_\mu j^\mu - L(x) \right) d^4 x.$$

The second pair of Maxwell equations should be calculated in assumption that the trajectory of the charge is fixed, and that its variation, and, thereby, the variation of the current, are zero. Thus, the variation of the action takes the form

$$\begin{aligned} \delta S &= -\frac{1}{c} \int \left[ \frac{1}{c} j^\mu \delta A_\mu + (1 - \sigma \mathfrak{F}(x)) \delta \mathfrak{F}(x) \right] d\Omega, \\ \delta \mathfrak{F}(x) &= \frac{1}{2} F^{\mu\nu} \delta F_{\mu\nu} = -\frac{1}{2} F^{\mu\nu} \left( \frac{\delta \partial A_\mu}{\partial x^\nu} - \frac{\delta \partial A_\nu}{\partial x^\mu} \right) = -F^{\mu\nu} \frac{\partial}{\partial x^\nu} \delta A_\mu. \end{aligned} \quad (2)$$

After integrating by parts and taking into account that there is no field at infinities the latter equation takes the form

$$\delta S = -\frac{1}{c} \int \left[ \frac{1}{c} j^\mu + \frac{\partial}{\partial x^\nu} [(1 - \sigma \mathfrak{F}(x)) F^{\mu\nu}] \right] \delta A_\mu d\Omega \quad (3)$$

As always, variations should turn to zero for any  $\delta A_\mu$ , and we come to the nonlinear field equation

$$\frac{\partial}{\partial x^\nu} [(1 - \sigma \mathfrak{F}(x)) F^{\mu\nu}] = -\frac{1}{c} j^\mu. \quad (4)$$

### III. FIELDS OF A MOVING CHARGE

#### A. Current

In this paper we restrict ourselves only to those currents  $j^\mu$  that correspond to a pointlike or any other charge, which is at rest in the origin in a certain Lorentz reference frame (the rest frame) and is distributed in a spherically symmetric and time-independent way in that frame. The charge moves as a whole with the speed  $-\mathbf{v}$ ,  $v < 1$ , along axis 1,  $v_i = \delta_{i1}v$ , in the inertial frame that moves relative to the rest frame with the 4-velocity  $u_\mu$ , which is the unit four-vector,  $u^2 = 1$ , with the components  $u_0 = \gamma$ ,  $u_i = \gamma \frac{v_i}{c}$ , where  $\gamma = (1 - v^2/c^2)^{-1/2}$ . The four-current is

$$j^\mu = u_\mu \rho(x), \quad (5)$$

where  $\rho(x)$  is the Lorentz scalar defined as the time-independent charge density in its rest frame. This implies that under Lorentz boosts and spacial rotations only the argument of  $\rho$  is transformed.

To make sure of the validity of the four-vector representation (5) let us first define the current in the rest frame as  $j'^0(x') = c\rho(\mathbf{x}')$ ,  $\mathbf{j}'(x') = 0$ . All quantities relating to that frame are marked with primes throughout. Applying the Lorentz transformation to this current

$$j_0 = \gamma \left( j'_0 + \frac{v}{c} j'_1 \right), \quad j_1 = \gamma \left( j'_1 + \frac{v}{c} j'_0 \right), \quad j_2 = j'_2, \quad j_3 = j'_3, \quad (6)$$

with the account of the vanishing of its spacial component in the rest frame  $j'_1 = 0$ , and applying the inverse transformation to the coordinates

$$x'_0 = \gamma \left( x_0 - \frac{v}{c} x_1 \right), \quad x'_1 = \gamma \left( x_1 - \frac{v}{c} x_0 \right), \quad x'_2 = x_2, \quad x'_3 = x_3. \quad (7)$$

we get for the moving charge (6)  $j^0(x) = \gamma c \rho((x_1 - \frac{v}{c} x_0) \gamma, x_2, x_3)$ ,  $\mathbf{j}(x) = \mathbf{v} \gamma \rho((x_1 - \frac{v}{c} x_0) \gamma, x_2, x_3)$ . This agrees with Eq. (5). For the pointlike charge  $e$  we have

$$\begin{aligned} j'^0(x') &= c\rho(\mathbf{x}') = ce\delta^3(\mathbf{x}'), & \mathbf{j}'(x') &= 0 \\ j^0(x) &= \gamma ce\delta\left(\left(x_1 - \frac{v}{c}x_0\right)\gamma\right)\delta(x_2)\delta(x_3) = ce\delta\left(x_1 - \frac{v}{c}x_0\right)\delta(x_2)\delta(x_3) = \gamma ce\delta^3(\mathbf{x}'). \\ \mathbf{j}(x) &= \mathbf{v}\gamma e\delta\left(\left(x_1 - \frac{v}{c}x_0\right)\gamma\right)\delta(x_2)\delta(x_3) = \mathbf{v}e\delta\left(x_1 - \frac{v}{c}x_0\right)\delta(x_2)\delta(x_3) = \mathbf{v}\gamma e\delta^3(\mathbf{x}'). \end{aligned} \quad (8)$$

Note that our final expressions for the current components differ by the Lorentz-factor  $\gamma$  from the corresponding expressions in Ref. [4], because the charge density  $\rho(x)$  is defined there as the zero-component of the four-current, and not as a Lorentz scalar, like here. The both ways are equivalent, of course.

Later we shall see that the current (8) is reproduced as the right-hand side of the field equation (4) for a moving point charge in nonlinear, as well as linear, electrodynamics, and also in the weak continuity equation (37) for the energy-momentum tensor.

#### B. Covariant ansatz

Once the motion of the charge is given, there are no other vectors in the problem besides  $u_\mu$  and the four-coordinate of the observation point  $x_\mu$ , which is the argument of the differential equations (4). Therefore, the potential produced by the charge may have the only representation

$$A^\mu = u^\mu f_1(xu, x^2) + x^\mu f_2(xu, x^2), \quad (9)$$

where  $f_1$  and  $f_2$  are functions of the indicated scalars. Then the field-strength tensor resulting from this potential is

$$F_{\mu\nu} = (u_\mu x_\nu - u_\nu x_\mu) f(xu, x^2) \quad (10)$$

with

$$f(xu, x^2) = \frac{\partial f_2}{\partial(xu)} - 2 \frac{\partial f_1}{\partial(x^2)}. \quad (11)$$

The requirement that in the rest frame the (electric) field  $F_{0i}$  be independent of time  $x_0$  implies that  $f$  should be a function of the combination

$$W^2 = (xu)^2 - x^2 \quad (12)$$

of its argument. So, by definition, the invariant  $W$  is the distance from the observation point to the charge in the rest frame of the latter  $W = |\mathbf{x}'| = r$ . (In the rest frame it is evident that Eq. (12) defines a positive quantity  $W = |\mathbf{x}'| = r$ . As long as  $W^2$  is a Lorentz invariant it remains positive in any frame.) Therefore, in what follows we shall keep to the representation

$$f(xu, x^2) = g(W^2). \quad (13)$$

It can be directly verified that the field tensor resulting from (10) and (13)

$$F_{\mu\nu} = (u_\mu x_\nu - u_\nu x_\mu) g(W^2) \quad (14)$$

satisfies the first pair of the Maxwell equations (the Bianchi identities)

$$\epsilon^{\mu\nu\lambda\rho} \frac{\partial F_{\nu\lambda}}{\partial x^\rho} = 0, \quad \epsilon^{0123} = 1. \quad (15)$$

However, it is easier to see this if we note that by choosing  $f_2 = 0$  in (9) and taking  $f_1$  as

$$f_1(W^2) = \frac{1}{2} \int g(W^2) dW^2$$

we determine the vector-potential generating the field (14), (13)

$$A_\mu = u_\mu f_1(W^2)$$

thus guaranteeing the fulfillment of (15). In the rest frame this potential is the 3-scalar  $A'_0 = f_1(\mathbf{x}'^2)$ ,  $\mathbf{A}' = 0$ . Setting  $f_2 = 0$  is not the only way to fix the vector potential generating Eq. (14). Another choice of the potential admitted within the gauge arbitrariness may be, for instance,

$$A_\mu = (u_\mu (ux) - x_\mu) g(W^2).$$

This potential satisfies the Lorentz-invariant gauge condition  $(Au) = 0$ .

In what follows we concentrate in finding a solution to the second pair of the nonlinear Maxwell equations (4) using the form (14) as an ansatz for searching for it.

### 1. Linear limit

Let us first note that the covariant extension of the Coulomb field produced by a moving point charge  $e$ , which is at rest in the origin in the rest frame, i.e., by the one, whose world line passes through the point  $\tilde{x}_0 = \tilde{x}_i = 0$ , is:

$$F_{\mu\nu}^{\text{lin}} = \frac{(u_\mu x_\nu - u_\nu x_\mu)}{W} \frac{e}{4\pi W^2}. \quad (16)$$

This corresponds to setting

$$g^{\text{lin}}(W^2) = \frac{e}{4\pi W^3} \quad (17)$$

in (14) in the linear limit. In the rest frame  $u_i = 0$ ,  $u_0 = 1$ , we have  $W = |\mathbf{x}'| = r$ , and then the nonvanishing components of Eq.(16) constitute the Coulomb electric field of a point-like charge:

$$F_{0i}^{\text{lin}} = \frac{x'_i}{r} \frac{e}{4\pi r^2}. \quad (18)$$

Expression (16) satisfies, in the moving frame, the linear Maxwell equation

$$\partial^\mu F_{\mu\nu}^{\text{lin}} = \frac{1}{c} j_\nu \quad (19)$$

with the current (8) of a point charge. This rather evident statement is explicitly demonstrated in V.

In order to reproduce the standard form of the electric and magnetic field components produced by a charge moving with the time-independent speed  $\dot{u}_\mu = 0$  with its worldline  $\tilde{\mathbf{x}} = \frac{\mathbf{v}}{c} \tilde{x}_0$  passing through the origin  $\tilde{\mathbf{x}} = 0$  at zero time-moment  $\tilde{x}_0 = 0$ , written in Ref. [4] in the case of linear Maxwell electrodynamics as

$$\mathbf{E}^{\text{lin}} = \left(1 - \frac{v^2}{c^2}\right) \frac{e(\mathbf{x} - \frac{\mathbf{v}}{c} x_0)}{4\pi R^{*3}}, \quad \mathbf{B}^{\text{lin}} = [\mathbf{v} \times \mathbf{E}^{\text{lin}}], \quad (20)$$

it is sufficient to note that  $R^*$  defined in [4] (when  $\mathbf{v}$  has only the first component,  $v_i = \delta_{i1}v$ ) as

$$R^{*2} = \left(x_1 - \frac{v}{c} x_0\right)^2 + (x_2^2 + x_3^2) \left(1 - \frac{v^2}{c^2}\right) \quad (21)$$

is related to the Lorentz scalar  $W$  (12) in the following way  $R^{*2} = \left(1 - \frac{v^2}{c^2}\right) W^2$  (Note that  $R'$  of Ref. [4] is just our invariant  $W$ ). With this substitution Eq. (20) is just what follows for the electric and magnetic components of (16).

The electric and magnetic fields (20) or (16) can be written also as functions of the difference  $\Delta x_\mu = x_\mu - \tilde{x}_\mu$  between the coordinate of the observation point  $x_\mu$  and that of the charge  $\tilde{x}_\mu$  as follows

$$\begin{aligned} E_i^{\text{lin}}(\Delta x, \tilde{x}) &= F^{\text{lin}0i} = \frac{e}{4\pi W^2} \frac{(u^0 \Delta x^i - u^i \Delta x^0)}{W}, \\ B_i^{\text{lin}}(\Delta x, \tilde{x}) &= \epsilon_{ijk} F_{jk}^{\text{lin}} = \epsilon_{ijk} \frac{e}{4\pi W^2} \frac{(u_j \Delta x_k - u_k \Delta x_j)}{W} \end{aligned} \quad (22)$$

with

$$W^2 = \left(1 - \frac{v^2}{c^2}\right)^{-1} \left[ \left(\Delta x_1 - \frac{v}{c} \Delta x_0\right)^2 + \left((\Delta x_2)^2 + (\Delta x_3)^2\right) \left(1 - \frac{v^2}{c^2}\right) \right] \quad (23)$$

As long as the trajectory is fixed, neither (20), nor (22) contain the variable  $\tilde{x}_\mu$  explicitly.

In order to write the fields in the Liénard-Wiechert form it is necessary to exploit the light-cone condition  $(\Delta x)^2 = 0$ , i.e.  $(\Delta x_2)^2 + (\Delta x_3)^2 = (\Delta x_0)^2 - (\Delta x_1)^2$  in (22), which tells that the charge and the observation point must be separated by a light-like interval for the field produced by the charge be nonzero in the observation point. Imposing the light-cone condition turns (23) into

$$W^2 = \left(1 - \frac{v^2}{c^2}\right)^{-1} \left(\Delta x_0 - \frac{v}{c} \Delta x_1\right)^2,$$

and (22) becomes Eqs. (63,8), (63,9) of Ref. [4] with  $\dot{\mathbf{v}} = 0$  after the identification  $R = \Delta x_0$ ,  $R_i = \Delta x_i$ .

In this connection, the following remark is in order. In linear electrodynamics the influence of a charge propagates with the speed of light in the vacuum  $c$ , and hence the observation point must be separated from the four-position of the source by a light-like interval,  $(\Delta x)^2 = 0$ , other space-time points carrying no field produced by the source at this position. This is not the case in nonlinear electrodynamics. It is well known already in QED that the nonlinearity leads [9] to nontrivial dielectric permeability and magnetic susceptibility of the vacuum in an external field, thereby to deviation of the speed of propagation from that of light. The role of the external field is in our case played by the field produced by the charge itself, the propagation speed depending, as a matter of fact, on its intensity. For this reason it will be not relevant to impose the light-cone condition onto the nonlinear solution of the next section for getting an analog of the Liénard-Wiechert potential.

### C. Solution to nonlinear field equations

From (4) we can conclude that its solution may be related to  $F^{\text{lin}\mu\nu}$  given by (16) as

$$(1 - \sigma \mathfrak{F}(x))F^{\mu\nu} = F^{\text{lin}\mu\nu}, \quad (24)$$

since (16) satisfies the linear Maxwell equation (19) as demonstrated in V. Then, using the ansatz (14) and taking into account that  $u^2 = 1$ , we get

$$\begin{aligned} (1 - \sigma \mathfrak{F}(x))g &= \frac{e}{4\pi W^3}, \\ \mathfrak{F}(x) &= \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}W^2g^2. \end{aligned} \quad (25)$$

This is the cubic equation for  $g(W^2)$ :

$$g^3 + \frac{2g}{\sigma W^2} - \frac{2e}{4\pi\sigma W^5} = 0. \quad (26)$$

The discriminant of this equation is positive

$$Q = \left(\frac{2}{3\sigma W^2}\right)^3 + \left(\frac{e}{4\pi\sigma W^5}\right)^2 > 0, \quad (27)$$

because  $W^2 > 0$ , as argued before. The only real Cardano solution for the case is

$$g = \sqrt[3]{\frac{e}{4\pi\sigma W^5} + \sqrt{Q}} + \sqrt[3]{\frac{e}{4\pi\sigma W^5} - \sqrt{Q}}. \quad (28)$$

Its asymptotic behavior near the charge

$$g \sim \left(\frac{2e}{4\pi\sigma W^5}\right)^{1/3} \quad \text{as } W \rightarrow 0, \quad (29)$$

will provide convergence for the field mass integral (42) in Subsection "Field mass". In the rest frame, when substituted into (14) with  $u_0 = 1$ ,  $u_i = 0$ ,  $W = r$  this reproduces the result of Ref. [2]. In the limit of vanishing nonlinearity  $\sigma \rightarrow 0$  Eq. (28) becomes  $g = \frac{e}{4\pi W^3}$ , therefore the solution Eq. (14) turns into expression (16) for the known field of a moving charge in the linear Maxwell electrodynamics,  $F_{\mu\nu} = F_{\mu\nu}^{\text{lin}}$ .

Finally, from (14) for the point charge moving along the trajectory  $\tilde{\mathbf{x}} = \frac{v}{c}\tilde{x}_0$  with a constant speed along axis 1, expressions for the electric and magnetic fields  $E_i = F^{0i}$  and  $B_i = \epsilon_{ijk}F_{jk}$  as functions of the observation coordinates are

$$\mathbf{E} = (1 - \frac{v^2}{c^2})^{-1/2}(\mathbf{x} - \frac{\mathbf{v}}{c}x_0)g(W^2), \quad \mathbf{B} = [\mathbf{v} \times \mathbf{E}]. \quad (30)$$

In terms of the distance from the charge  $\Delta x_\mu = x_\mu - \tilde{x}_\mu$  these are

$$\begin{aligned} \mathbf{E} &= \left(1 - \frac{v^2}{c^2}\right)^{-1/2} g(W^2) \left(\Delta \mathbf{x} - \frac{\mathbf{v}}{c} \Delta x_0\right), \\ B_1 &= 0, \quad B_2 = -2\frac{v}{c} \left(1 - \frac{v^2}{c^2}\right)^{-1/2} g(W^2) \Delta x_3, \quad B_3(\Delta x) = 2\frac{v}{c} \left(1 - \frac{v^2}{c^2}\right)^{-1/2} g(W^2) \Delta x_2. \end{aligned} \quad (31)$$

where  $W^2$  is the same as (23), and  $g(W^2)$  is the solution (28). This is the nonlinear generalization of Eqs. (20) and (22).

#### IV. SOLITONIC REPRESENTATION OF THE POINT-CHARGE FIELD

##### A. Energy-momentum tensor

In this section we set  $c = 1$ , and also  $\hbar = 1$  in the expression for  $\sigma$ . The Noether current corresponding to space-time translations

$$T^{\mu\nu} = \frac{\partial A^\lambda}{\partial x_\mu} \frac{\partial L(x)}{\partial (\partial A^\lambda / \partial x_\nu)} - g^{\mu\nu} L(x),$$

calculated using the Lagrange density (1) with the account of the field equations (4) is written as

$$T^{\mu\nu} = -[1 - \sigma \mathfrak{F}(x)] F^{\mu\lambda} F_\lambda^\nu - \frac{\partial}{\partial x_\lambda} [A^\mu (1 - \sigma \mathfrak{F}(x)) F_\lambda^\nu] - g^{\mu\nu} L(x). \quad (32)$$

This is the energy-momentum tensor of the electromagnetic field. The requirement that it should be gauge-invariant makes us omit the potential-dependent second term (that is a full derivative) to be left with the expression, symmetrical under the permutation of the indices:

$$T^{\mu\nu} = -[1 - \sigma \mathfrak{F}(x)] F^{\mu\lambda} F_\lambda^\nu - g^{\mu\nu} L(x). \quad (33)$$

Densities of the energy  $T^{00}$  and of the Pointing vector  $T^{i0}$  have the form

$$\begin{aligned} T^{00} &= -F^{0\lambda} F_\lambda^0 (1 - \sigma \mathfrak{F}(x)) + \mathfrak{F}(x) - \frac{\sigma}{2} (\mathfrak{F}(x))^2, \\ T^{i0} &= -F^{i\lambda} F_\lambda^0 (1 - \sigma \mathfrak{F}(x)). \end{aligned} \quad (34)$$

Contravariant components of the tensor (33) are

$$T_v^\mu = -[1 - \sigma \mathfrak{F}(x)] F^{\mu\lambda} F_{v\lambda} - \delta_v^\mu L(x). \quad (35)$$

Its trace is different from zero

$$T_i^i = 2\sigma (\mathfrak{F}(x))^2. \quad (36)$$

in the nonlinear case  $\sigma \neq 0$ .

##### 1. Weak continuity and energy-momentum conservation

The above construction of the energy-momentum tensor does not provide the fulfillment of its continuity, since the current is meant to be supported by outer forces and therefore we should admit the energy-momentum non-conservation. Instead of the continuity equation

we formally obtain what may be referred to as the "partial conservation of the Noether current" in the form

$$\frac{\partial}{\partial x^\mu} T_\nu^\mu = -F_{\nu\lambda} j^\lambda. \quad (37)$$

To derive this relation, equation of motion (4) was used in Appendix 2 together with the Bianchi identities  $\frac{\partial F_{\lambda\rho}}{\partial x^\nu} = -\frac{\partial F_{\rho\nu}}{\partial x^\lambda} - \frac{\partial F_{\nu\lambda}}{\partial x^\rho}$ . This relation is not owing to the nonlinearity and retains its form in the standard linear theory as well. The right-hand side in (37) is not vanishing. However the continuity of the energy-momentum tensor holds in a weak form

$$u^\nu \frac{\partial}{\partial x^\mu} T_\nu^\mu = 0. \quad (38)$$

To see this, note that the 4-current of a point charge moving with the constant speed is parallel to its 4-velocity vector. Then the right-hand side of (37) disappears when contracted with the 4-velocity,  $u^\nu F_{\nu\lambda} j^\lambda \sim u^\nu F_{\nu\lambda} u^\lambda = 0$ . The property of weak continuity (38) will be sufficient for establishing the conservation of the energy-momentum vector.

Let us define the latter by the integral over a space-like hyperplane that is orthogonal to the four-velocity and crosses the time axis at  $x_0 = s\sqrt{1-v^2}$

$$P_\mu = \int u^\nu T_\nu^\mu \delta(ux - s) d^4x. \quad (39)$$

Via the Gauss theorem it follows from the vanishing of the 4-divergence (38) that  $P_\mu$  is independent of  $s$ , and thereby of the time of observation  $x_0$ , because the hyperplane can be shifted as a whole along the vector  $u^\nu$  without affecting the value of the integral, since the fields (14) decrease at space-time infinity no less fast as in the linear electrodynamics, the nonlinearity fades away far from the charge, where its fields are weak. On the contrary, one cannot change to a space-like hyperplane inclined differently than in (39) in the definition of the energy-momentum 4-vector due to the lack of the continuity law analogous to (38) with the unit vector  $u^\nu$  other than the 4-velocity.

When the field  $F_{\nu\lambda}$ , on which the energy-momentum tensor  $T_\nu^\mu$  depends, is that of a uniformly moving (or resting) point charge, the integral in (39) usually diverges and hence makes no sense. This is not the case in the nonlinear theory under consideration here, as we shall see in the next subsection. Therefore, we may treat the energy-momentum vector (39) seriously.

Bearing in mind that  $s$  is a Lorentz scalar (moreover, set equal to zero in what follows), and that  $T_\nu^\mu$  (33) is a tensor, the integral (39) does define a Minkowski vector. It cannot help being directed along  $u^\nu$ , since this is the only external vector in the integrand of (39). Hence we write

$$P^\mu = u^\mu M_f, \quad (40)$$

where  $M_f$  is the field mass.

## B. Finite field mass

It follows from (39) that

$$M_f = P^\nu u_\nu = \int u_\nu T^{\mu\nu} u_\mu \delta(ux) d^4x.$$

From (33) and (16) we calculate

$$\begin{aligned} M_f &= \int \left[ -[1 - \sigma \mathfrak{F}(x)] u_\mu F^{\mu\lambda} F_\lambda^\nu u_\nu - L \right] \delta(ux) d^4x = \\ &= \int \left[ \left( 1 - \sigma \frac{W^2 g^2}{2} \right) W^2 g^2 + \frac{W^2 g^2}{2} + \frac{\sigma W^4 g^4}{8} \right] \delta(ux) d^4x. \end{aligned} \quad (41)$$



Thanks to the delta-function we may set  $W^2 = -x^2$  in the integrand, the argument of the function  $g(W^2)$  included. For any function  $\Phi(x^2)$ , provided the integrals below converge, the following chain of relations holds

$$\int \Phi(x^2) \delta(u x) d^4 x = \frac{1}{u_0} \int \Phi(-x_1^2 \frac{u_0^2 - u_1^2}{u_0^2} - x_2^2 - x_3^2) dx_1 dx_2 dx_3.$$

Here we have integrated over  $x_0$  using the delta-function. Performing the change of the variable  $x_1 \frac{\sqrt{u_0^2 - u_1^2}}{u_0} = x_1 \sqrt{1 - v^2} = x'_1$  and omitting the prime afterwards we find that this integral is equal to

$$\int \Phi(-\mathbf{x}^2) d^3 x.$$

Applying this result to (41) we get for the field mass the integral over the space

$$M_f = \int \left[ \frac{\mathbf{x}^2 g(\mathbf{x}^2)^2}{2} + \frac{3}{8} \sigma \mathbf{x}^4 g(\mathbf{x}^2)^4 \right] d^3 x.$$

Note that the function  $g(\mathbf{x}^2)$  involved here is just (28) taken in the rest frame.

With the help of Eqs.(26) the second term can be expressed as

$$\frac{3}{8} \sigma \mathbf{x}^4 g(\mathbf{x}^2)^4 = \frac{3}{4} \left( -\mathbf{x}^2 g(\mathbf{x}^2)^2 + \frac{e g(\mathbf{x}^2)}{4\pi |\mathbf{x}|} \right).$$

Then we can rewrite expression for the mass as

$$M_f = \int \left[ -\frac{\mathbf{x}^2 g(\mathbf{x}^2)^2}{4} + \frac{3}{4} \frac{e g(\mathbf{x}^2)}{4\pi |\mathbf{x}|} \right] d^3 x. \quad (42)$$

Therefore, to calculate the mass  $M_f$ , we have to calculate two integrals. The first one is

$$\begin{aligned} \int \mathbf{x}^2 g(\mathbf{x}^2)^2 d^3 x &= |e|^{\frac{3}{2}} \left( \frac{3}{2\sigma(4\pi)^2} \right)^{\frac{1}{4}} \frac{3}{2} I_1, \\ I_1 &= \int_0^\infty y^{\frac{2}{3}} \left( \sqrt[3]{\sqrt{1+y^4}+1} - \sqrt[3]{\sqrt{1+y^4}-1} \right)^2 dy = 0.885, \end{aligned} \quad (43)$$

and the second one is

$$\begin{aligned} e \int \frac{g(\mathbf{x}^2)}{4\pi |\mathbf{x}|} d^3 x &= |e|^{\frac{3}{2}} \left( \frac{3}{2\sigma(4\pi)^2} \right)^{\frac{1}{4}} I_2, \\ I_2 &= \int_0^\infty y^{-\frac{2}{3}} \left( \sqrt[3]{\sqrt{1+y^4}+1} - \sqrt[3]{\sqrt{1+y^4}-1} \right)^2 dy = 3.984. \end{aligned} \quad (44)$$

The final result is

$$M_f = |e|^{\frac{3}{2}} \left( \frac{3}{2\sigma(4\pi)^2} \right)^{\frac{1}{4}} \frac{1}{4} \left( 3I_2 - \frac{3}{2} I_1 \right) = 2.65 |e|^{\frac{3}{2}} \left( \frac{3}{2\sigma(4\pi)^2} \right)^{\frac{1}{4}} < \infty,$$

and we can see that this is the same value as the full electrostatic energy of a stationary pointlike particle found in [2].

## V. CONCLUSION

The object of the study in this paper is a nonlinear electrodynamics. We represented a moving pointlike electric charge as a soliton, a particle-like solution for its electromagnetic field that makes up a field configuration with finite energy. The property of finiteness of the field mass is inherent in many nonlinear models, for example in ones in Refs. [10], and in the first place, in the famous Born-Infeld model [11]. This property has been encouraging attempts to attribute the experimental value of the electron mass to the energy carried by its field, the electromagnetic contribution being sometimes almost exhausting [12].

To specialize our consideration, we have chosen the model of Ref. [2]. This nonlinear model originates from truncation of quantum electrodynamics at the second power of the field invariant  $\mathfrak{F}$ , and it may be thought of as the simplest model among those that result in finiteness of the field energy of a point charge, because it admits analytical solution to the field equations. Besides, its Lagrangian (1) does not possess the disadvantage of being a singular function of the field. It is this quality that led to presence of a maximum value of electric field, responsible, in the end, for the finiteness of the field energy in the Born-Infeld [11], and other [10], [12], like models. On the opposite, in the model of [2] the field of a point charge is still singular near the charge (29), but this singularity is suppressed to the extent sufficient for the convergence of the field-mass integral (41).

By explicitly solving the nonlinear Maxwell equation (4) in an inertial Lorentz reference frame with the use of covariant ansatz (14) that includes the 4-vector  $u_\mu$  of the particle speed, we found the electric and magnetic fields of a uniformly moving charge as functions of the observation point (30) and of the 4-distance to the moving charge (31).

We defined a gauge-invariant symmetric energy-momentum tensor (33) for the field of a moving charge. Irrespective of a special choice of nonlinear model and already in the linear limit  $\sigma = 0$ , the energy-momentum tensor possesses but a weak conservation property (37), (38), because the current, corresponding to the charge moving with a constant speed is external. We were able to establish, nevertheless, the time-independence of the energy and momentum as components of the Minkowskian vector defined as an integral (39) of the energy-momentum tensor over a hyperplane orthogonal to  $u_\mu$ . The energy-momentum vector (40) is the same as the mechanical energy-momentum of a mere massive particle (soliton) with the mass equal to the finite rest energy of the electric field.

The consideration in the present article can be readily extended to include any local nonlinear electrodynamics with convergent field energy. This condition is met if the corresponding Lagrangian in place of (1) grows with the field invariant already as  $\mathfrak{F}^w$  with  $w > \frac{3}{2}$  (to be published elsewhere).

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## Appendix 1

Here we demonstrate that Eq.(16) satisfies linear equation (19) with the point-like charge current (8). We omit the superscript “lin” within this Appendix.

It is convenient to rewrite (16) in terms of coordinates of the coordinate system  $K'$ , which moves along the axis  $x_1$  with constant velocity  $v$  (rest frame of the moving particle). The

corresponding Lorentz boost is,

$$x_0 = \gamma \left( x'_0 + \frac{v}{c} x'_1 \right), \quad x_1 = \gamma \left( x'_1 + \frac{v}{c} x'_0 \right), \quad x'_2 = x_2, \quad x'_3 = x_3.$$

and the inverse transformation is given by (7). The Lorentz transformation reduces the scalar  $W^2$  to the form

$$\begin{aligned} W^2 &= (xu)^2 - x^2 = \\ &= x_1'^2 + x_2'^2 + x_3'^2 = r'^2. \end{aligned} \quad (45)$$

The relations

$$\begin{aligned} \left. \frac{\partial x'_0}{\partial x_0} \right|_x &= \gamma \left. \frac{\partial (x_0 - \frac{v}{c} x_1)}{\partial x_0} \right|_x = \gamma, \\ \left. \frac{\partial x'_1}{\partial x_0} \right|_x &= \gamma \left. \frac{\partial (x_1 - \frac{v}{c} x_0)}{\partial x_0} \right|_x = -\gamma \frac{v}{c}, \\ \left. \frac{\partial x'_0}{\partial x_1} \right|_x &= \gamma \left. \frac{\partial (x_0 - \frac{v}{c} x_1)}{\partial x_1} \right|_x = -\gamma \frac{v}{c}, \\ \left. \frac{\partial x'_1}{\partial x_1} \right|_x &= \gamma \left. \frac{\partial (x_1 - \frac{v}{c} x_0)}{\partial x_1} \right|_x = \gamma, \end{aligned} \quad (46)$$

where  $|_x$  designates derivative calculated at constant  $x$ , follow from (7). Then

$$\begin{aligned} \left. \frac{\partial}{\partial x_0} \right|_x &= \left( \gamma \left. \frac{\partial}{\partial x'_0} \right|_{x'} - \gamma \frac{v}{c} \left. \frac{\partial}{\partial x'_1} \right|_{x'} \right), \\ \left. \frac{\partial}{\partial x_1} \right|_x &= \left( -\gamma \frac{v}{c} \left. \frac{\partial}{\partial x'_0} \right|_{x'} + \gamma \left. \frac{\partial}{\partial x'_1} \right|_{x'} \right). \end{aligned} \quad (47)$$

Now, with the account of (45)-(47), one can calculate divergency of  $F_{\mu\nu}$  exploiting the covariant form (16). The zeroth and first components are:

$$\begin{aligned} \partial^\mu F_{\mu 0} &= \frac{e}{4\pi} \left[ \frac{\partial}{\partial x_1} \left( \frac{-\gamma (x_1 - \frac{v}{c} x_0)}{W^3} \right) + \frac{\partial}{\partial x_2} \left( \frac{-\gamma x_2}{W^3} \right) + \frac{\partial}{\partial x_3} \left( \frac{-\gamma x_3}{W^3} \right) \right] = \\ &= \frac{e\gamma}{4\pi} \left[ \frac{\partial}{\partial x'_i} \left( \frac{-x'_i}{r'^3} \right) \right], \\ \partial^\mu F_{\mu 1} &= \frac{e}{4\pi} \left[ \frac{\partial}{\partial x_0} \left( \frac{\gamma (x_1 - \frac{v}{c} x_0)}{W^3} \right) + \frac{\partial}{\partial x_2} \left( \frac{-\gamma \frac{v}{c} x_2}{W^3} \right) + \frac{\partial}{\partial x_3} \left( \frac{-\gamma \frac{v}{c} x_3}{W^3} \right) \right] = \\ &= \frac{v}{c} \frac{e\gamma}{4\pi} \left[ \frac{\partial}{\partial x'_i} \left( \frac{-x'_i}{r'^3} \right) \right], \end{aligned} \quad (48)$$

The second and third components are

$$\partial^\mu F_{\mu 2,3} = \frac{e\gamma x_{2,3}}{4\pi} \left[ \frac{\partial}{\partial x_0} \left( \frac{1}{W^3} \right) + \frac{v}{c} \frac{\partial}{\partial x_1} \left( \frac{1}{W^3} \right) \right] = \frac{e\gamma x'_{2,3}}{4\pi} \left[ \frac{\partial}{\partial x_0} + \frac{v}{c} \frac{\partial}{\partial x_1} \right] \left( \frac{1}{r'^3} \right).$$

This is zero, because

$$\left[ \frac{\partial}{\partial x_0} + \frac{v}{c} \frac{\partial}{\partial x_1} \right] r'^{-3} = \left[ -\gamma \frac{v}{c} \frac{\partial}{\partial x'_1} + \gamma \frac{v}{c} \frac{\partial}{\partial x'_1} \right] r'^{-3} \equiv 0, \quad \left[ \frac{\partial r'^{-3}}{\partial x'_0} \right]_{x'=const} = 0.$$

Referring to the linear Maxwell equation in the rest frame

$$\frac{\partial}{\partial x'_i} \left( \frac{-x'_i}{r'^3} \right) = 4\pi\delta^3(\mathbf{x}'), \quad (49)$$

we can write

$$\partial^\mu F_{\mu\nu} = 4\pi \left( e\gamma\delta^3(\mathbf{x}'), \frac{v}{c}e\gamma\delta^3(\mathbf{x}'), 0, 0 \right).$$

This coincides with the current (8).

## Appendix 2

In this Appendix we present a detailed derivation of the partial conservation law Eq. (37) for a more general Lagrangian than (1)

$$L(x) = -\mathfrak{F}(x) + \mathfrak{L}, \quad \mathfrak{L} = \mathfrak{L}(\mathfrak{F}(x)), \quad (50)$$

where  $\mathfrak{L}$  is an arbitrary function of  $\mathfrak{F}(x)$ . The second pair of nonlinear Maxwell equations in place of (4) now reads

$$\frac{\partial}{\partial x^\nu} \left[ \left( 1 - \frac{\partial \mathfrak{L}}{\partial \mathfrak{F}(x)} \right) F^{\mu\nu} \right] = -\frac{1}{c} j^\mu, \quad (51)$$

and the stress-energy tensor (33) becomes

$$T^{\mu\nu} = -F^{\mu\lambda} F_\lambda^\nu \left( 1 - \frac{\partial \mathfrak{L}}{\partial \mathfrak{F}(x)} \right) + g^{\mu\nu} \mathfrak{F}(x) - g^{\mu\nu} \mathfrak{L}. \quad (52)$$

To obtain the partial continuity equation (37) one needs to calculate the derivative  $\frac{\partial T^{\mu\nu}}{\partial x^\mu}$  on solutions of the field equations, i.e. by using (51) and the Bianchi identities (15). We have

$$\frac{\partial}{\partial x^\mu} \left[ - \left( 1 - \frac{\partial \mathfrak{L}}{\partial \mathfrak{F}(x)} \right) F^{\mu\lambda} F_{\nu\lambda} \right] = -\frac{1}{c} j^\lambda F_{\nu\lambda} - \left( 1 - \frac{\partial \mathfrak{L}}{\partial \mathfrak{F}(x)} \right) F^{\mu\lambda} \frac{\partial F_{\nu\lambda}}{\partial x^\mu}, \quad (53)$$

where

$$-\delta_\nu^\mu \frac{\partial}{\partial x^\mu} [-\mathfrak{F}(x) + \mathfrak{L}] = \left( 1 - \frac{\partial \mathfrak{L}}{\partial \mathfrak{F}(x)} \right) \frac{\partial \mathfrak{F}(x)}{\partial x^\nu}. \quad (54)$$

Let us calculate the derivative

$$\frac{\partial \mathfrak{F}(x)}{\partial x^\nu} = \frac{1}{2} F^{\lambda\rho} \frac{\partial F_{\lambda\rho}}{\partial x^\nu}. \quad (55)$$

Next we use (15) to transform the derivative  $\frac{\partial F_{\lambda\rho}}{\partial x^\nu}$  as

$$\frac{\partial F_{\lambda\rho}}{\partial x^\nu} = -\frac{\partial F_{\rho\nu}}{\partial x^\lambda} - \frac{\partial F_{\nu\lambda}}{\partial x^\rho}.$$

Now Eq. (54) can be written as

$$-\delta_\nu^\mu \frac{\partial}{\partial x^\mu} [-\mathfrak{F}(x) + \mathfrak{L}] = \left( 1 - \frac{\partial \mathfrak{L}}{\partial \mathfrak{F}(x)} \right) \frac{1}{2} F^{\lambda\rho} \left( -\frac{\partial F_{\rho\nu}}{\partial x^\lambda} - \frac{\partial F_{\nu\lambda}}{\partial x^\rho} \right). \quad (56)$$

Taking into account that  $F_{\mu\nu}$  is antisymmetric, and making the change of the summation indices  $\lambda \rightarrow \mu$  in the first term, and  $\rho \rightarrow \mu$  in the second, we can see that the both terms in

the latter expression coincide and their sum is just the second term in (53) with the opposite sign to cancel its contribution into the full derivative of  $T_\nu^\mu$ . Gathering together (53) and (56) we finally obtain (37).

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